Higher order topological actions

Roman V. Buniy[1,*] and Thomas W. Kephart[2,†]

[1] Institute of Theoretical Science, University of Oregon, Eugene, OR 94703

[2] Department of Physics and Astronomy,
Vanderbilt University, Nashville, TN 37235

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In classical mechanics, an action is defined only modulo additive terms which do not modify the equations of motion; in certain cases, these terms are topological quantities. We construct an infinite sequence of higher order topological actions and argue that they play a role in quantum mechanics, and hence can be accessed experimentally.

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I. INTRODUCTION

Measurable phases in physics, from those appearing in high energy scattering to those on which superconducting devices are based, have played a critical role in applying our understanding of quantum mechanics. These phases typically arise in interference phenomena, and can be topological or geometric. Of the topological phases, only those related to first order (Gaussian) linking have been studied in detail. However, there is an infinite set of higher order topological linkings, and in this paper we argue that this higher order set has a concomitant infinite set of phases, all in principle detectable in the laboratory. While the proper treatment of this topic is necessarily somewhat mathematical, the results are physically predictive and imminently testable.

An equation of motion of a dynamical system is a stationary point of an action. As a result, different actions can lead to the same equation of motion. In particular, addition of a quantity S to the action does not change the equation of motion if $\delta S = 0$. As an example, consider $S = \int_C A$, where C is an oriented curve and A is a differential 1-form [1]. The variation is expressed in terms of the Lie derivative, $\delta S = \int_C \pounds_{\delta x} A$. Since $\delta x|_{\partial C} = 0$, the condition $\delta S = 0$ leads to $dA = 0$.

The condition $dA = 0$ makes a set of closed curves C special since small deformations of such C do not change the value of S. In such a case, S depends only on global properties of C and consequently it is a topological quantity. Hereafter we consider only closed curves C and call S a topological term.

If A is exact, then $S = 0$, trivially. Hence we are interested in closed 1-forms which are not exact and therefore physically relavent. Let A^* denote the vector space of such forms. We will show that $A^* = \cup_{p\geq 1} A^{(p)}$, where each space $A^{(p)}$ is constructed from spaces $A^{(q)}$, where $q < p$. The space $A^{(1)}$ is generated by the elements of the first cohomology group $H^1(M)$. If M is simply connected, then $H^1(M)$ is trivial and the topological term vanishes. If M is non-simply connected, then $H^1(M)$ is nontrivial and the topological term can be nonzero. All elements of $A^{(1)}$ are local quantities and all elements of $A^{(p)}$ for $p \geq 2$ are

*Electronic address: roman@uoregon.edu
†Electronic address: tom.kephart@gmail.com

nonlocal quantities; the degree of nonlocality increases with p.

For a given C, there is an associated vector space of topological terms, $S^* = \int_C A^*$, where $S^* = \cup_{p\geq 1} S^{(p)}$ and $S^{(p)} = \int_C A^{(p)}$. A given closed curve C belongs to one of the homotopy classes which are the elements of the fundamental group $G = \pi_1(M)$; the value of $\int_C A$ is the same for all curves in a class.

In quantum mechanics, the elements of S^* should form abelian representations of the group of allowed curves. We will show that this leads to the set of subgroups $G_* = \{G_p\}_{p\geq 1}$ such that the elements of $S^{(p)}$ form abelian representations of G_p. This means that a topological term of only one order will occur in any action [2]. We then proceed to construct $A^{(p)}$ and G_p iteratively and show their relation to the homotopy classes of paths.

While there is no reason to study topological terms in classical dynamics, such terms are important in quantum dynamics. The Aharonov-Bohm effect is a famous example demonstrating importance of topological terms in quantum mechanics. We review why $S^{(1)}$ is responsible for this effect and then argue that the higher order spaces $S^{(p)}$, $p \geq 2$, can also lead to measurable effects in quantum-mechanical systems.

II. TOPOLOGICAL TERMS

Without loss of generality, as an example of a 3-dimensional non-simply connected space we take $M = \mathbb{R}^3 - T$, where $T = \cup_{1\leq i\leq N} T_i$ is the union of disjoint tubes. Each tube $T_i = C_i \times D_i$ is a direct product of a closed curve C_i and a disk D_i. Various topological properties of the space M can be deduced from its homology and cohomology groups [3]. The first homology group $H_1(M)$ is a group of closed curves modulo those which are boundaries of surfaces. The first cohomology group $H^1(M)$ is a group of closed 1-forms modulo exact forms. For the present case, the basis of $H_1(M)$ is $\{\partial D_i\}_{1\leq i\leq N}$ and the basis of $H^1(M)$ is $\{A_i\}_{1\leq i\leq N}$. By the de Rham theorem, the 1-forms can be chosen such that the two bases are dual to each other, $\int_{\partial D_i} A_j = \delta^i_j$. This duality condition cannot be uniquely solved for 1-forms; a convenient particular solution [4] is

$$A_i(x) = \int_{y\in\Sigma_i} \delta(x-y) \sum_{1\leq a\leq 3} dx^a * dy^a. \tag{1}$$

Here Σ_i is an oriented surface for which C_i is the boundary, and $*$ is the Hodge star operator. Since A_i is singular on Σ_i and vanishes everywhere else, a closed curve C which intersects Σ_i once in the positive direction contributes δ^i_j to the integral $\int_C A_j$; the duality condition follows. We define $A^{(1)}$ as a vector space with the basis $\{A_i\}_{1\leq i\leq N}$. For a given closed curve C, there is an associated vector space of first order topological terms $S^{(1)} = \int_C A^{(1)}$.

To define second order topological terms, consider $F_{ij} = A_i \wedge A_j$ for $i \neq j$. Modulo a constant factor, F_{ij} is a unique closed 2-form which can be expressed in terms of A_i and A_j. We define a 1-form A_{ij} by means of an equation $dA_{ij} = F_{ij}$. (If C_i and C_j are unlinked, Σ_i and Σ_j can be chosen to be disjoint, in which case $dA_{ij} = 0$.) A particular solution of this equation is

$$A_{ij} = \tfrac{1}{2}\gamma_i A_j - \tfrac{1}{2} A_i \gamma_j. \tag{2}$$

where $\gamma_i = \delta_i + \int_\Gamma A_i$ and δ_i is a constant. A path Γ is the part of C which starts at x_0 and ends at x; the orientations of C and Γ agree. We define $A^{(2)}$ as a vector space with the basis $\{A_{ij}\}_{1\leq i<j\leq N}$. For a given closed curve C, there is an associated vector space of second order topological terms, $S^{(2)} = \int_C A^{(2)}$.

To proceed, we look for a closed 3-form F_{ijk} for $i \neq j \neq k$ which can be expressed in terms of the corresponding elements of $A^{(1)}$ and $A^{(2)}$. Without loss of generality, we have

$$F_{ijk} = A_{ij} \wedge A_k + A_i \wedge A_{jk}. \tag{3}$$

Since $dF_{ijk} = 0$, we can define quantities A_{ijk} by means of equations $dA_{ijk} = F_{ijk}$. (If the first and second order linkings for (C_i, C_j, C_k) vanish, then $(\Sigma_i, \Sigma_j, \Sigma_k)$ can be chosen to be disjoint, in which case $dA_{ijk} = 0$.) Particular solutions of these equations are

$$A_{ijk} = \gamma_{ij} A_k - A_i \gamma_{jk}, \tag{4}$$

where $\gamma_{ij} = \delta_{ij} + \int_\Gamma A_{ij}$ and δ_{ij} is a constant. We define $A^{(3)}$ a vector space with the basis $\{A_{ijk}\}_{i\neq j\neq k}$. For a given closed curve C, there is an associated vector space of third order topological terms, $S^{(3)} = \int_C A^{(3)}$.

It is clear how to construct higher order topological terms. The vector spaces $\{A^{(p)}\}$ are related to what is known in algebraic topology as the Massey products of cohomology groups [5]; see also [6].

III. RESTRICTIONS

In the previous section, the spaces $A^{(p)}$ were defined only on M. We now extend these definitions into the interiors of the tubes T. Such extensions are always possible if certain topological restrictions are satisfied. It turns out that in order to define the space $A^{(p)}$, all spaces $A^{(q)}$ with $q < p$ have to be defined. If we assume that all $A^{(q)}$ with $q < p$ are defined, then we denote $R^{(p)}$ a set of additional restrictions needed to define the space $A^{(p)}$. We now find $R^{(p)}$ iteratively.

No restrictions are needed to define $A^{(1)}$; this means $R^{(1)} = \varnothing$. To find $R^{(2)}$, consider extending A_{ij} inside T_i for $i \neq j$. This extension is possible only if $dF_{ij} = 0$ inside T_i, which means that $\int_{\partial T_i} F_{ij} = 0$. However, since

$$\int_{\partial T_i} A_i \wedge A_j = \int_{T_i} d(A_i \wedge A_j) = \int_{C_i} A_j, \tag{5}$$

there is an obstruction to such a procedure unless $\int_{C_i} A_j = 0$. No new restriction is needed to extend A_{ij} inside T_j. Therefore, $A^{(2)}$ can be defined only if a set of restrictions

$$R^{(2)} = \left\{ \int_{C_i} A_j = 0 \right\}_{i\neq j} \tag{6}$$

is satisfied. This means that all pairs of distinct loops (C_i, C_j) should be unlinked. To find $R^{(3)}$, consider extending A_{ijk} inside T_i, T_j, and T_k for $i \neq j \neq k$. Reasoning as above, we find that $A^{(3)}$ can be defined only if a set of restrictions

$$R^{(3)} = \left\{ \int_{C_i} A_{jk} = 0 \right\}_{i\neq j\neq k} \tag{7}$$

is satisfied. This means that the second order linking between any triple of distinct loops (C_i, C_j, C_k) should vanish.

It is clear how to proceed to construct higher order restrictions $R^{(p)}$, $4 \leq p \leq N$. In order to construct all spaces $\{A^{(p)}\}_{1\leq p\leq N}$, the set of curves $\{C_i\}$ has to satisfy the restrictions $R = \cup_{1\leq p\leq N} R^{(p)}$. From the property that in order for $S^{(p)}$ to be defined, all $S^{(q)}$ with $q < p$ have to be defined, we see that $R' = \cup_{1\leq p\leq N'} R^{(p)}$ is always satisfied for some $N' \leq N$. In this case, all linkings of orders $2 \leq p \leq N'$ for N curves $\{C_i\}$ vanish. For $N' = N$, the set of curves is unlinked through Nth order. [7] (As a curious observation, note that this simplest topological arrangement of loops provides the richest structure for the topological term.)

IV. COMPUTATION

To derive explicit expressions for the elements of $S^{(p)}$, it is convenient to proceed as follows. First note [3] that for $M = \mathbb{R}^3 - T$, a manifold with N unlinked tubes removed, the fundamental group is $G = \pi_1(M) = \mathbb{Z} * \cdots * \mathbb{Z}$, the free product [9] of N copies of $\mathbb{Z}$. This group is of infinite order and it is freely generated by a set of generators $\{a_i\}_{1\leq i\leq N}$. (These generators are homotopically equivalent to $\{\partial D_i\}_{1\leq i\leq N}$.) A generator a_i is defined as a closed path in M, which starts at the point x_0, intersects Σ_i once in the positive direction, does not intersect any other Σ_j, $j \neq i$, and ends at x_0. The inverse path a_i^{-1} is the path a_i traversed in the opposite direction. To multiply paths, we compose them in such a way such that the end of the previous path is the beginning of the next path. Homotopy classes of paths are labeled by finite sets of integers $(n_{11}, \ldots, n_{N1}, \ldots, n_{1l}, \ldots, n_{Nl})$, and representative paths from such classes are given by

$$C = a_1^{n_{11}} \cdots a_N^{n_{N1}} \cdots a_1^{n_{1l}} \cdots a_N^{n_{Nl}}. \tag{8}$$

For the topological terms $S_i = \int_C A_i$, $S_{ij} = \int_C A_{ij}$, $S_{ijk} = \int_C A_{ijk}$ we find

$$S_i = \sum_{i'} n_{ii'}, \tag{9}$$

$$2S_{ij} = \delta_i S_j - S_i \delta_j + \sum_{i'j'} \sigma_{i'j'} n_{ii'} n_{jj'}, \tag{10}$$

$$4S_{ijk} = \delta_i S_{jk} - S_i \delta_j S_k - S_{ij}\delta_k - \delta_i S_j \delta_k + 2\delta_{ij} S_k - 2 S_i \delta_{jk} + \sum_{i'j'k'} \sigma_{i'j'k'} n_{ii'} n_{jj'} n_{kk'}, \tag{11}$$

where $\sigma_{ij} = 1$ for $i \leq j$ and $\sigma_{ij} = -1$ for $i > j$, and $\sigma_{ijk} = 1$ for $i \leq j \leq k$ or $k+2 \leq j+1 \leq i$, and $\sigma_{ijk} = -1$ otherwise. Expressions for higher order topological terms are similarly found.

Elements of $S^{(1)}$ depend only on a path; as a result, they are additive for multiplicative paths, $S_i(CC') = S_i(C) + S_i(C')$. In other words, elements of $S^{(1)}$ form abelian representations of the group G. The situation is different for elements of $S^{(p)}$ for $p \geq 2$; they depend on both the path and the location of the point x_0 through constants $\{\delta_i\}$, $\{\delta_{ij}\}, \ldots$. Since the constants can be different for different loops in a product of loops, these topological terms are not in general additive for multiplicative paths, but below we show that there is a particular set of terms that are additive.

V. TOPOLOGICAL QUANTUM PHASES

Classical dynamics is determined by the path which extremizes the action. In quantum dynamics, all curves (paths) $C \in G$ contribute to an amplitude through the Feynman weight factor e^{iS}. This allows interference between topologically inequivalent terms and it means that although topological terms do not affect classical dynamics, they can affect quantum dynamics.

If the set of restrictions R' is satisfied, all spaces $\{S^{(p)}\}_{1\le p\le N'}$ can contribute to the phase of the wave function. This obviously presents a problem when $N' \ge 2$ since elements of $S^{(p)}$ for $p \ge 2$ do not form abelian representations of the group G. We solve this problem by constructing subgroups of G for which the abelian property of the topological terms holds. First note that $S^{(2)}$ is independent of x_0 only if $S^{(1)}$ is the zero vector space. It can be shown that in this case a closed curve C is a product of commutator loops. (A commutator loop [9] is a path $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$, where $g_i \in G$.) It is easy to verify that for the product of commutator loops an element of $S^{(2)}$ is the sum of the corresponding terms for each component, $S_{ij}(CC') = S_{ij}(C) + S_{ij}(C')$. This means that elements of $S^{(2)}$ form abelian representations of the subgroup $G_2 = [G, G]$ generated by commutators of elements of G. It is clear that it is enough to consider a path $C = [a_i^{n_i}, a_j^{n_j}]$, for which we find $S_{ij} = n_i n_j$.

Similarly, in order for the elements of $S^{(3)}$ to satisfy the abelian property, $S^{(1)}$ and $S^{(2)}$ must be zero vector spaces. In this case, the path C is a product of second order commutator loops $[g_1, [g_2, g_3]]$, where $g_i \in G$, and so elements of $S^{(3)}$ form abelian representations of the subgroup $G_3 = [G, G_2]$ generated by commutators of elements of G and G_2. The simplest second order commutator loop is $C = [a_i^{n_i}, [a_j^{n_j}, a_k^{n_k}]]$, for which we find $S_{ijk} = n_i n_j n_k$. Elsewhere [8], we will provide details and show how this procedure for higher order topological terms $S^{(p)}$ naturally leads to groups G_p which are known as the subgroups of the lower central series [9] of G.

According to a theorem [10] for path integrals in non-simply connected spaces, the phase of the wave function in quantum mechanics has to form an abelian representations of the fundamental group. By the above construction, higher order boundary terms can be included and the phase of order p is $\int_C A$, where $A \in A^{(p)}$ and $C \in G_p$.

Quantum mechanics imposes restrictions on what elements of $S^{(p)}$ are allowed to contribute to the phase. This can be seen as follows. If a charged particle is transported along a closed curve C outside a solenoid, then its action changes by $\int_C A$, where A is the gauge potential of the magnetic field in the solenoid. The Aharonov-Bohm effect [11] states that the wave function acquires a phase $\phi = \xi n \Phi$, where $\xi = e(\hbar c)^{-1}$, n is the number of times the curve wraps around the solenoid, and Φ is the flux of the magnetic field. To relate this to the calculation for $p = 1$ above, we take a path $C = a_i^{n_i}$ with the corresponding $S_i = n_i$, and find the first order phase $\phi_i = \xi S_i \Phi_i$. For $p = 2$, we take a path $C = [a_i^{n_i}, a_j^{n_j}]$ with the corresponding $S_{ij} = n_i n_j$, and find the second order phase $\phi_{ij} = K_2 \xi^2 S_{ij} \Phi_i \Phi_j$, where K_2 is a constant. Proceeding similarly, we find the phase of order p,

$$\phi_{i_1 \cdots i_p} = K_p \xi^p S_{i_1 \cdots i_p} \Phi_{i_1} \cdots \Phi_{i_p}, \tag{12}$$

where K_p is a constant. Except for $K_1 = 1$, constants K_p are undetermined [12]. We are not aware of any fundamental quantum-mechanical principle [13] forbidding the presence of terms with $p \ge 2$ and therefore suggest this be tested experimentally.

Let us assume for simplicity that all N loops are totally unlinked. For $N = 1$, only the usual Aharonov-Bohm term ϕ_1 can contribute to the phase of the wave function. For $N = 2$, the second order contribution ϕ_{12} is present if and only if both first order contributions ϕ_1 and ϕ_2 vanish; no higher order terms are present. The simplest generalization of the Aharonov-Bohm effect is provided by the path $C = [a_1, a_2]$. In Ref. [14] we proposed a test of the presence of this term in the wave function by suggesting an experimental setup to detect the phase which we calculated to be $\phi_{12} = K_2 \xi^2 \Phi_1 \Phi_2$.

We have considered the case when M is 3-dimensional. The corresponding construction for $d = 2$ can be easily obtained from the one for $d = 3$. Indeed, we can smoothly deform the tubes in such a way that a plane intersects each tube twice along a pair of disjoint disks; we then replace each curve C_i by a pair of points. Since 3-forms dF_{ij}, $dF_{ijk}, \ldots$ now vanish, there are no topological restrictions for definitions of the spaces $\{A^{(p)}\}_{1 \le p \le N}$. All other results are readily translated from the $d = 3$ case. We will study the case $d \ge 4$ elsewhere [8].

VI. CONCLUSIONS

The action of a system is not uniquely defined since arbitrary topological terms can be added to the action without changing the equation of motion. Although classical dynamics is immune to such terms, they affect the quantum dynamics. These terms can be classified according to their topological properties. Each term contributes a phase to the wave function, the functional form of which is easily distinguishable from the phases due to terms of other orders. In particular, the phase of order p is proportional to the product of p fluxes. The usual Aharonov-Bohm phase corresponds to $p = 1$, and its simplest generalization is the Borromean ring phase which corresponds to $p = 2$. Examples of higher order phases $\phi_{i_1 \cdots i_p}$ due to higher order linking [15], will correspond to general order p. It should not be difficult to conduct an experiment capable of answering the question whether higher order topological phases play a role in quantum mechanics.

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[1] We summarized here few relevant facts about differential forms. The exterior (wedge) product is anti-symmetric, $dx^a \wedge dx^b = -dx^b \wedge dx^a$. The exterior derivative of a 1-form $A = A_a dx^a$ is a 2-form $dA = (\partial A_a / \partial x^b) dx^b \wedge dx^a$. The exterior product of two 1-forms is a 2-form, $A \wedge B = A_a B_b dx^a \wedge dx^b$. Higher order forms are obtained by repeated use of operations of differentiation and multiplication. For a d-dimensional manifold M and $(d-1)$-form Q, the Stokes theorem states that $\int_{\partial M} Q = \int_M dQ$, where ∂M is the boundary of M.

[2] We could write a compound multisystem lagrangian where this is not the case, but for the discussion here we consider a minimal lagrangian where only one boundary term is allowed.

[3] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[4] In a gauge theory this corresponds to a choice of gauge. The reader might find it useful to view $\{A_i\}$ as gauge potentials of the magnetic fields confined to the tubes $\{T_i\}$, but the results in the text are independent of this analogy.

[5] W. S. Massey, Symp. Int. Topologia Algebraica, Mexico, 145 (1959); W. S. Massey, Proc. Conf. on Algebraic Topology, Chicago, University of Illinois at Chicago, p. 174 (1968); D. Kraines, Trans. Amer. Math. Soc. **124**, 431 (1966); E. J. O'Neill Trans. Amer. Math. Soc. **248**, 37 (1979); R. A. Fenn, *Techniques of Geometric Topology*, Cambridge University Press, Cambridge, 1983; M. I. Monastyrsky and V. S. Retakh, Commun. Math. Phys. **103**, 445 (1986).

[6] M. A. Berger, J. Phys. A, **23**, 2787 (1990).

[7] To be more precise, consider a tube T_i. If it has the pth order linking with the set of tubes $\{T_i\}_p' = \{T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_p\}$, then it cannot have lower order linking with any subset of these tubes. Furthermore, the fact that T_i has the pth order linking with $\{T_i\}_p'$ means that the set $\{T_i\}_p = \{T_1, \ldots, T_p\}$ cannot be involved in any qth order linking for $q > p$. The Whitehead link of two tubes has non-gaussian linking (order > 2).

[8] R. V. Buniy and T. W. Kephart, in preparation.

[9] W. Magnus, A. Karrass, D. Solitar, *Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations*, Interscience, New York, 1966.

[10] L. S. Schulman, J. Math. Phys. **12**, 304 (1971); M. G. G. Laidlaw and C. M. DeWitt, Phys. Rev. D **3**, 1375 (1971); L. S. Schulman, *Techniques and Applications of Path Integration*, Wiley-Interscience, New York, 1981.

[11] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).

[12] There is an argument allowing to calculate the constants K_p for $p \geq 2$. From the Aharonov-Bohm result, if $(2\pi)^{-1}\xi\mathit{\Phi}_i \in \mathbb{Z}$, then the phase ϕ_i is unobservable. If this is also the case for the higher order phases, then we find $K_p = (2\pi)^{-p+1}$. In any case, the values of K_p for $p \geq 2$ should be determined by an experiment.

[13] One possible objection that can be raised regarding the higher order terms is that all elements of $A^{(p)}$ for $p \geq 2$ are nonlocal quantities. After addition of these terms, the coordinate and momentum operators are still local, but the hamiltonian operator becomes nonlocal. This nonlocality, however, has no local consequences. (In the magnetic field analogy, the only measurable effect is the force acting on the particle and it is absent outside the tubes.) This is analogous to the first order term having no local consequences despite being the local operator itself. Elsewhere [8], we will elaborate on this matter.

[14] See R. V. Buniy and T. W. Kephart, [arXiv:hep-th/0612***], where a Borromean ring arrangement to detect the second order phase ϕ_{12} was suggested. Two magnetic solenoids T_1 and T_2 carrying flux $\mathit{\Phi}_1$ and $\mathit{\Phi}_2$, and a path C corresponds to the closed path $C = a_1 a_2 a_1^{-1} a_2^{-1}$ formed by two topologically distinct paths C' and C'' for the wave function of a charged particle start from the source and end at the screen. To prevent first order (gaussian) linking of the wave function with the solenoids one would install a rectangular plate in the plane of T_1 that covers the area between the two sides of T_1 and fills the region between the sides of T_2. For particle wave packets that do not spread much beyond the center of the region containing the plate, only second order phase $\phi_{12} = K_2\xi^2\mathit{\Phi}_1\mathit{\Phi}_2$ will be detected at the screen.

[15] Examples of higher order linking, at arbitrary order p, can be found in L. Kauffman, *Knots and physics*, World Scientific, Singapore, 2001.